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Coefficient inequalities for certain subclass of starlike function with respect to symmetric points related with q-exponential function

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Abstract

The purpose of the present work is to define a class of starlike function with respect to symmetrical points in the domain of exponential functions related to the $\boldsymbol{q}$-exponential functions. We will determine the possible upper bound of the third Hankel determinant for the function $\boldsymbol{q}$-starlike.

Keywords: Analytic function[1], Symmetric point[2], q-derivative[3], Subordination[4], Hankel Determinant[5]

1. Introduction

Analytic functions also defined as holomorphic function are complex valued functions that are defined and differentiable at every point within their domain of definition. The class of all analytic function f with the normalization conditions in the open unit disc $E = \{z: |z| < 1\}$ is symbolized by A and has the Taylor series we have

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \ . \tag{1}$$

The class of univalent and analytic function unit disc E is prove by S. Caratheodory functions are a class of complex valued function defined on a domain in the complex plane. They are named after the mathematician Carathéodory, denoted by p, and the functions of this class are of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots. \tag{2}$$

The Schwarz function, named after the German mathematician Hermann Schwarz, is a complex-valued functions that maps the open disk E in the complex planes to itself. It is known by $f(z) = \frac{-z}{1-z^2}$ where z is a complex number. Specifically, if f and g are analytic function defined on some domain D, then $f \prec g$ if there lie other analytic function h defined on D such-that, $f(z) = g\big(h(z)\big)$ $\forall z$ in D. Thomas [1] and Pommerenke [2] defined the Hankel determinant $H_k(n)$, for positive integer k, n for the function in S of the form eq (1), as shown.

$$H_k(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix}. \quad (3)$$

For fixedly positive integer k and n the growths of $H_k(n)$ as $n \to \infty$. has been determined by Obradovic [3] in 2023, with a boundedness boundar. Ehrenborg [4] investigated the Hankel determinant for exponential polynomial. The Hankel determinant of different orders are achieved for different rates of *k, n*. For instance when *k = 2* and *n = 1* is defined as,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$$

$$= \left| a_1 a_3 - a_2^2 \right|, \; a_1 = 1 \quad (4)$$

The Fekete-Szego inequality is well-known results in complex analysis and potential theory that provides an estimate for the growth of the Taylor coefficients of function that are analytic unit disk *E*. More precisely, let $f\ (z)$ be function that are analytic in the open unit disc $|z| < 1$, and let its Taylor series expansions be given by (1). In 2023, Singh et al. [5] Fekete-Szego defined an inequality for the co-efficient of univalent analytic function on the unit disk. Let f be a univalent function then $\left| a_3 - \eta a_2^2 \right| \le 1 + 2\exp(\frac{-2\eta}{1-\eta}),\ 0 \le \eta < 1$. The study of the univalency of analytic functions also naturally give birth to the Fekete-Szego coefficient functional. Numerous authors have looked into the Fekete-Szego for function in different univalent subclasses, see [6]. Other researcher like Ali et al [7], [8], Owa and Cho [9], [10], Orhan & Cotirla [11], In 2007, Murugusundaramoorthy et al. [12] investigated the Fekete-Szego inequality for a number of normalized analytical functions $f(z)$ as known in the open unit disk E. In 2006, Shanmugam et al [13] introduced the Fekete Szegö problem can be applied to subclasses of starlike functions when considering symmetrical points. Now for $k = 2, n = 2$ it can be obtained that

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$$

$$= | \mathrm{a}_2 \mathrm{a}_4 - a_3^2 | . \quad (5)$$

In 2012, Krishna and Ramreddy [14] introduced the 2nd Hankel determinant of means the univalent function is discussed here. Using a indeed close *p*, valent function we calculate the growth rate of the 2nd Hankel determinant by Shrigan [15] In 2022. Other researchers like Janteng et al [16],[17], Bansal [18], Lee, et al [19], Lei et al [20],Rain et al [21], Rajya et al [22], Omer et al [46],Zaprawa [23] introduced for the coefficient of the function f that belong to the sub-class S of univalent function or to its sub-classes the upper-bound of the Hankel determinant for $k = 2, n = 3$ i.e., $H_2(3)$ is defined as:

$$H_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix}$$
$$= \left| a_3 a_5 - a_4{}^2 \right| \quad . \tag{6}$$

For $k = 3, n = 1$ the Hankel determinant, $H_3(1)$ is know as 3rd Hankel determinant we have

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} .$$

$$= a_3(a_2 a_4 - a_3{}^2) - a_4(a_1 a_4 - a_2 a_3) + a_5(a_3 - a_2{}^2), \quad a_1 = 1 \; . \tag{7}$$

q-calculus is a methodology equal to the use course of calculus but which is center on the solution of logic q-analogous result with-out the useless of limit. Lashin [42] deserves credit for the systematic introduction of q-calculus and Khanetal [43] introduced and provided definitions for q-derivative. q-derivative of function f be a normalized analytic function is known as

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)\,\mathrm{z}}, z \neq E \; .$$

and $D_q f(0) = f'(0)$ where $q \in (0,1)$ taking $q \to 1^-$ we get $D_q f \to f'$ In 2007, Babalola [24] was defined as the 1st person to analysis the upper-bound of the 3rd Hankel determinant for subclass of S. In 2016, Vamshee Krishna et al [25] introduced this study uses Toeplitz determinants to give the tops upper bound on the $H_3(1)$ Hankel determinant for starlike function with regard to the symmetrical point. In 2017, Patil and Khairnar [26] introduced using the Toeplitz determinant, the best achievable upper limit was found for the hankel-determinant for the starlike functions with regard to the symmetrical points. In 2017 Prajapat et al [27] investigated the Taylor coefficients of functions $f(z)$ belongings to a definite class of an analytical function in the open unit disk E are explored to determine the 3rd Hankel determinant. Other researchers like Yalcin and Altinkaya [28], Mohd Narzan et al. [32], Lecko et al [30], Kowalczyk et al [31], and Cho et al. [29]. Shi et al [43], Verma et al [44], Rath et al [45], Mahmood et al [37] investigated the class of univalent function starlike with respect to symmetrical points. In 2015, Mendiratta et al [33] studied the class of analytic function f in the open unit disc as S_e^* . These functions must be normalised such that $f(0) = f'(0) - 1 = 0$ and must also be able to satisfy the requirement $\frac{zf'(\mathrm{z})}{f(\mathrm{z})} \prec e^z$ when |z| < 1. It is possible to obtain the structural formula inclusion relation coefficient estimations, growth and distortion result subordination theorem and a variety of radii constants for function that belong to the class S_e^* . In 2018, HaiYan Zhang et al [34] investigated the class of analytical function $f(z)$ in the open, unit disc E normalized by $f(0) = 0$ and $f'(0) = 1$ which is subordinate to the exponential functions, is denoted by S_l^*. In 2022, Khan et al [35] studied the context of the sine function, We will describe a class of starlike function with regard to symmetrical locations in this article. In 2022, Senguttuvan et al [36] define A thorough sub-class of an analytic function w.r.t symmetrical point has been developed. We will expand the research using quantum calculus, and we will investigate the upper bounds of the

3rd Hankel Determinant, for the classes of a star-like function w.r.t symmetrical point subordinate to exponential functions. Now we defined the following sub-classes.

Definition1. A functions $f \in A$ and f to be in the class $S_s^*(e^{qz})$ as

$$\frac{2[zf'(z)]}{f(z)-f(-z)} \prec e^{qz}, z \in E \quad . \tag{8}$$

We note that taking $q \to 1^-$ in above definition, we obtain the known class $S_s^*(e^z)$, see [38].

The q-calculus is a methodology comparable to the usual study of calculus but which is centered on the idea of deriving q-analogous results without the use of limits. The main tool is the q-derivative. The q-derivative of a function f, which is a normalized analytic function, is defined as

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \,. (z \neq 0, 0 < q < 1).$$

And $D_q f(0) = f'(0)$ where $q \in (0,1)$. The following Lemmas are required to demonstrate the intended outcomes.

Lemma 2 [39] If $p \in P$, then $|p_n| \leq 2, \forall n \in N$.

Lemma 3 [40] If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$. is such that $Re(p(z)) > 0$ in E, then for some x, z with $|x| \leq 1, |z| \leq 1$, we have

$$2p_2 = p_1^2 + x(4 - p_1^2), \text{ for some } x, |x| \leq 1. \tag{9}$$

$$4p_3 = p_1^3 + 2p_1(4-p_1^2)x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z. \tag{10}$$

Lemma 4 [41] If $p \in P, then\ |p_2 - vp_1^2| \leq \max |\{1, |2v-1|\}$ for any $v \in C$.

2. Main Result

Theorem.5 If $f \in S_s^*(e^{qz})$ then $|a_2| \leq \frac{q}{2}, |a_3| \leq \frac{q}{2}, |a_4| \leq \left|\frac{q}{4} + \left(\frac{-4q+3q^2}{8}\right) + \left(\frac{12q-18q^2+5q^3}{48}\right)\right|, |a_5| \leq \left|\frac{q}{4} + \left(\frac{-2q+q^2}{4}\right) + \left(\frac{-6q+9q^2-3q^3+q^4}{24}\right)\right|.$

Proof: As $f \in S_s^*(e^{qz})$

$$\frac{2[zf'(z)]}{f(z)-f(-z)} = e^{qw(z)} \quad . \tag{11}$$

Using (1), we consider

$$\frac{2[zf'(z)]}{f(z)-f(-z)} = 1 + 2a_2 z + 2a_3 z^2 + (4a_4 z^4 - 2a_3 a_2) z^3 + (4a_5 z^5 - 2a^2{}_3) z^4 + \ldots \,. \tag{12}$$

Let us define the function

$$p(z) = \frac{1 + qw(z)}{1 - qw(z)}.$$

Equivalent,

$$qw(z) = \frac{p(z)-1}{p(z)+1}. \tag{13}$$

Consider

$$qw(z)=\frac{p_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{2}+\frac{p_1^3}{8}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_4}{2}-\frac{p_2^2}{4}+\frac{3p_1^2p_2}{8}-\frac{p_1^4}{16}\right)z^4+\cdots. \quad (14)$$

Since we have

$$e^{q\mathrm{w}(z)}=1+qw(\mathrm{z})+\frac{(qw(\mathrm{z}))^2}{2!}+\frac{(qw(\mathrm{z}))^3}{3!}+\frac{(qw(\mathrm{z}))^4}{4!}+\cdots . \quad (15)$$

We get

$$e^{qw(z)}=1+\frac{qp_1z}{2}+\left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)q+\left(\frac{p_1^2q^2}{8}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{4}+\frac{p_1^3}{48}\right)q+\left(\frac{-p_1^2q^2}{8}+\frac{p_1p_2q^2}{4}+\frac{p_1^3q^3}{48}\right)z^3+\left(\frac{p_4}{2}-\frac{p_1p_3}{4}-\frac{p_2^2}{8}+\frac{p_1^2p_2}{16}+\frac{p_1^4}{384}\right)q+\left(\frac{p_2^2}{8}+\frac{3p_1^4}{32}-\frac{p_1^2p_2}{4}+\frac{p_1p_3}{4}\right)q^2+\left(-\frac{p_1^2p_2}{16}-\frac{p_1^4}{32}\right)q^3+\left(\frac{p_1^4q^4}{384}\right)z^4+\cdots . \quad (16)$$

From eq (12) and eq (16) we compare the coefficient, we get

$$a_2=\frac{qp_1}{4},a_3=\left(\frac{p_2}{4}-\frac{p_1^2}{8}\right)q+\frac{q^2p_1^2}{16},a_4=\frac{p_3q}{8}-\frac{p_1p_2q}{8}+\frac{3p_1p_2q^2}{32}+\frac{p_1^3q}{32}-\frac{p_1^3q^2}{64}+\frac{5p_1^3q^3}{384},$$

$$a_5=\left(\frac{p_4q}{8}+\left(-\frac{p_1p_3q}{8}+\frac{p_1p_3q^2}{16}\right)+\left(\frac{3p_1^2p_2q}{32}-\frac{3p_1^2p_2}{32}-\frac{3p_1^2p_2q^2}{32}\right)+\left(\frac{p_1^4q^2}{128}-\frac{p_1^4q}{64}-\frac{p_1^4q^3}{128}+\frac{p_1^4q^4}{384}\right)\right. . \quad (17)$$

By using Lemma 2 and Lemma 4 in eq (17), we get

$$|a_2|\le\frac{q}{2},|a_3|\le\frac{q}{2},|a_4|\le\left|\frac{q}{4}+\left(\frac{-4q+3q^2}{8}\right)+\left(\frac{12q-18q^2+5q^3}{48}\right)\right|,$$

$$|a_5|\le\left|\frac{q}{4}+\left(\frac{-2q+q^2}{4}\right)+\left(\frac{-6q+9q^2-3q^3+q^4}{24}\right)\right|. \quad (18)$$

Which are the required results.

Theorem 6 If $f\in S_s^*(e^{qz})$ then $|a_3-a_2^2|\le\frac{q}{2}$.

Proof: From eq (17) in theorem 5 we have

$$a_2=\frac{qp_1}{4},a_3=\left(\frac{p_2}{4}-\frac{p_1^2}{8}\right)q+\frac{q^2p_1^2}{16}. \quad (19)$$

On simplifying, we get

$$|a_3-a_2^2|=\left|\frac{qp_2}{4}-\frac{q^2p_1^2}{8}\right|. \quad (20)$$

By using Lemma 4, we get

$$|a_3-a_2^2|\le\frac{q}{2}. \quad (21)$$

Which are the required result.

Theorem 7 If $f\in S_s^*(e^{qz})$ then $|a_2a_3-a_4|\le\frac{1}{48(q^2-6q-12)}(3q($

$$3q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q^4-36q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$q^3+6q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q^4+12q^5+2q^4\sqrt{-2q^3+8q^2+48q+64}$$

$$+36q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2 q^2-72q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)$$

$$q^3-144q^4-8\sqrt{-2q^3+8q^2+48q+64}\;q^3+432q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$q+72q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q^2+144q^3-48\sqrt{-2q^3+8q^2+48q+64}$$

$$q^2+432q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2+864q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)$$

$$q+1728q^2-16\sqrt{-2q^3+8q^2+48q+64}\;q+864q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)+1600q$$

Proof: From eq (17) of theorem 5, we have

$|a_2a_3-a_4| = \left|\frac{q^2p_1p_2}{16}-\frac{p_1^3q^2}{32}+\frac{q^3p_1^3}{64}-\frac{p_3q}{8}+\frac{p_1p_2q}{8}-\frac{3p_1p_2q^2}{32}-\frac{p_1^3q^2}{32}+\frac{p_1^3q^2}{64}-\frac{5p_1^3q^3}{384}\right|$. (22)

Denotes $|x| = t \in [0,1], p_1 = e \in [0,2]$, using triangle inequality eq (22) we have

$|a_2a_3-a_4| \le \frac{e(4-e^2)qt^2}{32}-\frac{t(4-e^2)q^2e}{64}-\frac{q(4-e^2)}{16}+\frac{e^3q^3}{384}$.

Suppose that

$$F(e,1) \equiv \frac{e(4-e^2)qt^2}{32}-\frac{t(4-e^2)q^2e}{64}-\frac{q(4-e^2)}{16}+\frac{e^3q^3}{384}.$$

Thus we get $\frac{\partial F}{\partial t} = \frac{(4-e^2)etq}{16}-\frac{(4-e^2)eq^2}{64} \ge 0$, the function *F(e,t)* is non-decreasing for any *t* in *[0,1]*.This show that *F(e,t)* has max value at *t=1*.

$$MaxF(e,t) = F(e,1) = \frac{e(4-e^2)qt^2}{32}-\frac{t(4-e^2)q^2e}{64}-\frac{q(4-e^2)}{16}+\frac{e^3q^3}{384}.$$

Which implies that

$$M(e) = \frac{e(4-e^2)qt^2}{32}-\frac{t(4-e^2)q^2e}{64}-\frac{q(4-e^2)}{16}+\frac{e^3q^3}{384}.$$

Then

$$M'(e) = \left(-\frac{qe^2}{16}+\frac{(-e^2+4)q}{32}-\frac{e^2q^2}{32}-\frac{(-e^2+4)q^2}{64}-\frac{eq}{8}+\frac{e^2q^3}{128}\right).$$

$M'(e) =$ be lost $e = m^* = -\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}$.

A simple computational yield that $M''(e) < 0$ which mean that the functions $M(e)$ can take max value at $m^* = -\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}$

Hence we get

$$|a_2a_3-a_4| \le \frac{1}{48(q^2-6q-12)}(3q(3q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)$$

$$q^4-36q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2 q^3+6q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)$$

$$q^4+12q^5+2q^4\sqrt{-2q^3+8q^2+48q+64}+36q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$q^2-72q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right) q^3-144q^4-8\sqrt{-2q^3+8q^2+48q+64}$$

$$q^3+432q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2 q+72q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)$$

$$q^2+144q^3-48\sqrt{-2q^3+8q^2+48q+64}\; q^2+432q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$+864q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right) q+1728q^2-16\sqrt{-2q^3+8q^2+48q+64}$$

$$q+864q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)+1600q\ . \qquad (23)$$

Which are the required results.

Theorem 8. If $f \in S_s^*(e^{qz})$ then $|a_2a_4-a_3^2| \le \frac{3q^2}{8}$.

Proof: From (17) of theorem 4, we have

$$|a_2a_4-a_3^2| = \left|\frac{p_1p_3q^2}{32}+\frac{3p_1^2p_3q^3}{128}-\frac{p_1^2p_3q^2}{32}+\frac{p_1^4q^2}{128}-\frac{3p_1^4q^3}{256}+\frac{5p_1^4q^4}{1536}-\frac{p_1^2p_2}{128}-\frac{p_1^4}{1536}-\frac{p_2^2q^2}{16}-\frac{p_2p_1^2q^3}{32}+\frac{p_2p_1^2q^2}{16}\right|.$$

We use the Lemma 4 we have

$$|a_2a_4-a_3^2| = \left|\frac{p_1q^2(4-p_1^2)(1-|x|^2)z}{64}-\frac{p_1^2(4-p_2^2)x^2q^2}{128}+\frac{3p_1^2q^3x(4-p_1^2)}{256}-\frac{x^2q^2(4-p_1^2)^2}{64}+\frac{5p_1^4q^4}{1536}-\frac{p_1^4q^2}{64}+\frac{p_1^4q^2}{32}-\frac{p_1^4q^4}{64}-\frac{p_1^4q^4}{256}\right|. \qquad (24)$$

Denotes $|x| = t \in [0,1], p_1 = e \in [0,2]$ then using triangle inequality we have

$$|a_2a_4-a_3^2| \le \frac{q^2(4-e^2)}{32}+\frac{e^2(4-e^2)t^2q^2}{128}+\frac{3e^2q^3t(4-e^2)}{256}+\frac{t^2q^2(4-e^2)^2}{64}+\left(\frac{5e^4q^4}{1536}-\frac{e^4q^2}{64}+\frac{e^4q^2}{32}-\frac{e^4q^4}{64}-\frac{e^4q^4}{256}\right). \qquad (25)$$

Which implies that

$$F(e,t)=\frac{q^2(4-e^2)}{32}+\frac{e^2(4-e^2)t^2q^2}{128}+\frac{3e^2q^3t(4-e^2)}{256}+\frac{t^2q^2(4-e^2)^2}{64}+\left(\frac{5e^4q^4}{1536}-\frac{e^4q^2}{64}+\frac{e^4q^2}{32}-\frac{e^4q^4}{64}-\frac{e^4q^4}{256}\right). \tag{26}$$

Thus we get

$$\frac{\partial F}{\partial t}=\frac{3(4-e^2)e^2q^2}{256}+\frac{(4-e^2)e^2q^2}{64}+\frac{(4-e^2)^2q^2t}{32}+\frac{(4-e^2)e^2q^2t}{64}\geq 0.$$

Which gives that *F(e, t)* is increasing for any then *t* in *[0,1]*, this show that *F (e, t)* has maxi value at *t = 1*.

$$Max\,F(e,t)=F(e,t)=\frac{q^2(4-e^2)}{32}+\frac{e^2(4-e^2)t^2q^2}{128}+\frac{3e^2q^3t(4-e^2)}{256}+\frac{t^2q^2(4-e^2)^2}{64}+\left(\frac{5e^4q^4}{1536}-\frac{e^4q^2}{64}+\frac{e^4q^2}{32}-\frac{e^4q^4}{64}-\frac{e^4q^4}{256}\right). \tag{27}$$

Let us define

$$M(c)=\frac{q^2(4-e^2)}{32}+\frac{e^2(4-e^2)t^2q^2}{128}+\frac{3e^2q^3t(4-e^2)}{256}+\frac{t^2q^2(4-e^2)^2}{64}+\left(\frac{5e^4q^4}{1536}-\frac{e^4q^2}{64}+\frac{e^4q^2}{32}-\frac{e^4q^4}{64}-\frac{e^4q^4}{256}\right). \tag{28}$$

We have

$$M'(e)=\frac{5e^3q^2}{128}-\frac{(-5e^2+4)eq^2}{128}-\frac{(eq^2)e}{16}-\frac{25e^3q^4}{384}.$$

$M'(e)$ vanishes at *e = 0*. A simple compilation yield that *M ''(e) < 0*, which mean that the functions *M (e)* has maximum-values at *e = 0*. we get

$$|a_2a_4-a_3^2|\leq M(0)=\frac{3q^2}{8}. \tag{29}$$

Which are the required results.

Theorem 9 If $f\in S_s^*(e^{qz})$ *then* $|H_3(1)=\frac{1}{2304(q^2-6q-12)^2}(q^3(15$

$$q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2 q^4-180q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$q^3+30q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q^4+10q^4\sqrt{-2q^3+8q^2+48q+64}$$

$$+60q^5+180q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$q^2-360q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q^3-40\sqrt{-2q^3+8q^2+48q+64}$$

$$q^3-288q^4+2160q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$q+360q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q^2-240\sqrt{-2q^3+8q^2+48q+64}$$

$$q^2-4464q^3+2160q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2+$$

$$4320q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q-80\sqrt{-2q^3+8q^2+48q+64}$$

$$q+13824q^2+4320q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)$$

$$+70208q+62208))+\frac{1}{2}\left(\frac{1}{4}q-\frac{1}{8}q^2-\frac{1}{8}q^3+\frac{1}{24}q^4\right)q \quad . \quad (30)$$

Proof: $H_3(1)=a_3(a_2a_4-a^2{}_3)-a_4(a_1a_4-a_2a_3)+a_5(a_3-a^2{}_2)$, $a_1=1$. (31)

By applying triangle triangle inequality, we get

Now, substituting the equation (18), (21), (23), (29), (30) in (31) we get

$$|H_3(1)=\frac{1}{2304(q^2-6q-12)^2}(q^3(15\,q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$q^4-180q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$q^3+30q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q^4+10q^4\sqrt{-2q^3+8q^2+48q+64}$$

$$+60q^5+180q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$q^2-360q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q^3-40\sqrt{-2q^3+8q^2+48q+64}$$

$$q^3-288q^4+2160q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2$$

$$q+360q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q^2-240\sqrt{-2q^3+8q^2+48q+64}$$

$$q^2-4464q^3+2160q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)^2+$$

$$4320q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)q-80\sqrt{-2q^3+8q^2+48q+64}$$

$$q+13824q^2+4320q\left(-\frac{2\left(-4+\sqrt{-2q^3+8q^2+48q+64}\right)}{q^2-6q-12}\right)$$

$$+70208q+62208))+\frac{1}{2}\left(\frac{1}{4}q-\frac{1}{8}q^2-\frac{1}{8}q^3+\frac{1}{24}q^4\right)q. \quad (32)$$

Which are the required results.

Remark 10. We note that on taking $q \to 1^-$ for the result proved in this research article, we obtain the results already proved in [38].

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